Chaos in Symmetric Phase Oscillator Networks

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Phase-coupled oscillators serve as paradigmatic models of networks of weakly interacting oscillatory units in physics and biology. The order parameter which quantifies synchronization so far has been found to be chaotic only in systems with inhomogeneities. Here we show that even symmetric systems of identical oscillators may not only exhibit chaotic dynamics, but also chaotically fluctuating order parameters. Our findings imply that neither inhomogeneities nor amplitude variations are necessary to obtain chaos; i.e., nonlinear interactions of phases give rise to the necessary instabilities.

Introduction.—Models of coupled oscillators describe various collective phenomena in natural and artificial systems, including the synchronized flashing of fireflies, the dynamics of superconducting Josephson junctions, oscillatory neural activity, and oscillations in chemical reaction kinetics [1]. In particular, phase-coupled models serve as paradigmatic approximations for many weakly coupled limit cycle oscillators [2,3]. The Kuramoto model (and its extensions) provides the gold standard in this field because it suitably describes the dynamics of a variety of real systems, is extensively studied numerically and reasonably understood analytically [4,5]. Each oscillator k with phase $\varphi_k(t) \in \mathbb{R}/2\pi\mathbb{Z} =: T$ on the 1-torus T changes with time t according to

$$\frac{d\varphi_k}{dt} = \omega_k + \frac{1}{N}\sum_{j=1}^{N} g(\varphi_k - \varphi_j) \qquad (1)$$

for all $k \in \{1, \ldots, N\}$. For the original Kuramoto model the coupling function g has a single Fourier mode, $g = \sin$. The dimension of such systems can be reduced to low dimensions [6–8], implying dynamics that is either periodic or quasiperiodic. For coupling functions with two or more Fourier components the collective dynamics may be much more complicated. For example, stable heteroclinic switching may emerge [9,10]. More irregular, chaotic dynamics of system (1) is observed for nonidentical oscillators only [11,12], raising the question whether inhomogeneities are necessary for the occurrence of such dynamics. To the best of our knowledge, the only hint that chaotic dynamics might exist for symmetric phase-oscillator networks are attractors with irregular structure in phase space found recently in a system of $N = 5$ oscillators [9].

The complex order parameter

$$R(t) = \frac{1}{N}\sum_{j=1}^{N} \exp(i\varphi_j(t)) \in \mathbb{C} \qquad (2)$$

where $i = \sqrt{-1}$ constitutes an important characteristic for coupled oscillator systems. In particular, its absolute value $|R(t)|$ quantifies their degree of synchrony with $|R(t)| = 1$ if all oscillators are in phase. For the original Kuramoto system the full complex order parameter (2) acts as a mean field variable enabling closed-form analysis [13].

In homogeneous, globally coupled systems (1) it remains unknown whether or not there exists any coupling function g that gives rise to chaotic order parameter fluctuations. Synchronous solutions, antisynchronous splay states as well as the dynamics of cluster states (where $\varphi_k = \varphi_j$ for at least one $k \neq j$) all yield a periodic complex order parameter $R(t)$. For any invariant solutions on tori, $R(t)$ is either periodic or quasiperiodic. The most irregular dynamics of $R(t)$ observed so far is due to heteroclinic cycles where $R(t)$ is nonperiodic as it "slows down" each cycle. Even if chaotic dynamics does emerge within the system, it may average out due to symmetry, possibly resulting in a regular dynamics of the order parameter.

In this Letter, we answer the question whether chaotic dynamics, and moreover, chaotic order parameter fluctuations may arise for some g even in the absence of inhomogeneities in (1). We show that indeed chaos is not possible for $N < 4$. By contrast, for $N = 4$, chaotic attractors can appear in a specific family of coupling functions g. Interestingly, attractors of all theoretically possible symmetries exist and we provide further examples of attracting chaos for $N = 5$ and $N = 7$. The existence of chaos for infinite families of $N \geq 4$ implies that chaos occurs in systems with certain $N > N_0$ for any $N_0 \in \mathbb{N}$ and suggests that chaos is likely to occur in many high-dimensional systems with a suitable choice of g.

No chaos for $N = 2$ and $N = 3$.—Let T^N be the N-dimensional torus and let S_N denote the group of permutations of N symbols. Suppose M is a differentiable manifold and let Γ be a group that acts on M. Recall that a vector field X on M is called Γ equivariant if X

"commutes" with the action of Γ, i.e., $X \circ \gamma = \hat{\gamma} \circ X$ for all $\gamma \in \Gamma$ where $\hat{\gamma}$ denotes the induced action on the tangent space.

Equivariance implies restrictions of the dynamics specified by the vector field. We study the dynamical system on T^N given by the ordinary differential equations (1). Let us henceforth assume that the system is homogeneous, i.e., $\omega_k = \omega$ for all $k \in \{1, \ldots, N\}$. This system is $S_N \times T^1$ equivariant where S_N acts by permuting indices and T^1 through a phase shift. Recall some basic properties of this system [2]. Introducing phase differences $\psi_j := \varphi_j - \varphi_1$ for all $j \in \{1, \ldots, N\}$ eliminates the phase-shift symmetry. Write $\varphi = (\varphi_1, \ldots, \varphi_N)$ and $\psi = (\psi_1, \ldots, \psi_N)$. The reduced system on T^{N-1} is given by

$$\dot{\psi}_j = \frac{1}{N}\left(\sum_{k=1}^{N} g(\psi_j - \psi_k) - \sum_{k=1}^{N} g(-\psi_k)\right) \quad (3)$$

for all $j \in \{2, \ldots, N\}$.

For any partition $P = \{P_1, \ldots, P_m\}$ of $\{1, \ldots, N\}$ (that is $P_r \subset \{1, \ldots, N\}$, $\bigcup_{j=1}^{m} P_j = \{1, \ldots, N\}$, and $P_r \cap P_s = \varnothing$ for $r \neq s$) the subspaces

$$F_P := \{\varphi | j, k \in P_r \text{ for any } r \Rightarrow \varphi_j = \varphi_k\} \subset T^N \quad (4)$$

are flow invariant. The subspaces divide T^{N-1} in $(N-1)!$ invariant $(N-1)$-dimensional simplices [2]; one of which

$$\mathcal{C} := \{\psi | 0 = \psi_1 < \psi_2 \cdots < \psi_N < 2\pi\} \subset T^{N-1} \quad (5)$$

we refer to as the *canonical invariant region*. There is a $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ symmetry on the canonical invariant region and the "splay state" (the phase-locked state with $\psi_j = 2\pi j/N$) is the only fixed point of this action at the centroid of this region.

The reduction of symmetry has implications for the existence of chaos in low dimensions. For $N = 2$ and $N = 3$ the phase space of the reduced system is a one-, resp. two-dimensional torus. This means that by the Poincaré-Bendixon theorem [14] chaos is not possible in these systems for $N < 4$.

Chaos and symmetry for $N = 4$.—We choose a parametrization of the coupling function g in (1) by considering a truncated Fourier series

$$g(\varphi) = \sum_{k=1}^{4} a_k \cos(k\varphi + \xi_k). \quad (6)$$

In particular, we restrict ourselves to the two parameter family given by the parametrization

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (\eta_1, -\eta_1, \eta_1 + \eta_2, \eta_1 + \eta_2) \quad (7)$$

where η_1 and η_2 are real valued parameters and $a_1 = -2$, $a_2 = -2$, $a_3 = -1$, and $a_4 = -0.88$ are constants.

For $N = 4$, chaotic attractors do indeed exist. The dynamics of the absolute value of the order parameter exhibits chaotic fluctuations and exponential divergence of trajectories, cf. Fig. 1. To explore parameter space, we
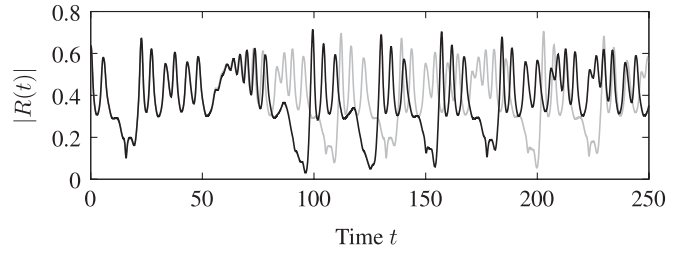

FIG. 1. Trajectories of the absolute value of the order parameter $|R(t)|$ fluctuate chaotically ($N = 4$). Two trajectories with small difference in initial condition diverge from each other (coupling function (7) for parameter values $\eta_1 = 0.1104$ and $\eta_2 = 0.5586$).

calculated the maximal Lyapunov exponent $\lambda_{\max}$ from the variational equations [15]. There are regions in (η_1, η_2)-parameter space in which $\lambda_{\max}$ is greater than zero, cf. Fig. 2. As might be expected, there is fine structure in this region, for example, islands where the trajectory converges to a stable limit cycle. Lines of period doubling cascades [16] bound the chaotic region and end in a homoclinic flip bifurcation with an inclination flip [17] (details not shown). Exploring initial conditions revealed the coexistence of chaotic attractors and stable limit cycles in part of the chaotic region.

Chaotic attractors in equivariant dynamical system can exhibit symmetries themselves. Let A be a chaotic attractor as defined in [18], i.e., a Lyapunov-stable, closed, and connected set that is the ω-limit set of a trajectory, for a dynamical system on a manifold M given by a Γ-equivariant vector field. The subgroup $\text{Stab}(A) := \{\gamma \in \Gamma | \gamma(a) = a \text{ for all } a \in A\}$ is the group of instantaneous symmetries of the attractor; i.e., at any point in time the action of $\text{Stab}(A)$ keeps every point in A fixed. Furthermore, we define $\Sigma(A) := \{\gamma \in \Gamma | \gamma(A) = A\}$ to be

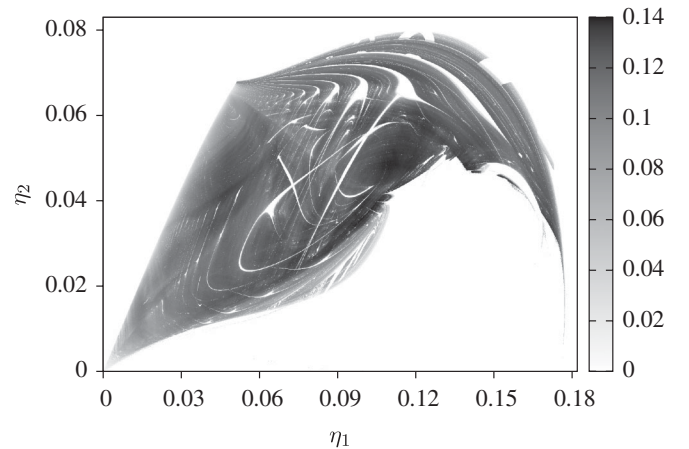

FIG. 2. Chaos for $N = 4$. Maximal Lypunov exponents are positive in a region of parameter space. The initial condition was fixed and the coupling function parametrized by (7).

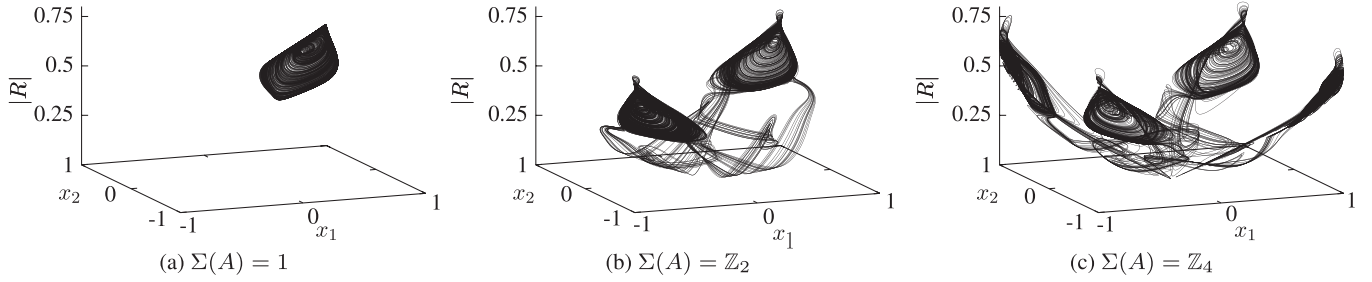(a) $\Sigma(A) = 1$ (b) $\Sigma(A) = \mathbb{Z}_2$ (c) $\Sigma(A) = \mathbb{Z}_4$

FIG. 3. All possible symmetries of the chaotic attractors for different parameters for $N = 4$. We have $\eta_1 = 0.138$ in panel (a), $\eta_1 = 0.0598$ in panel (b), $\eta_1 = 0.1104$ in panel (c), and $\eta_2 = 0.5586$ in all panels. The projection is a $\Gamma = S_4$ equivariant map, $x_1 = \sin(\varphi_2 - \varphi_4)$, $x_2 = \sin(\varphi_1 - \varphi_3)$, and $x_3 = |R|$.

the set of symmetries on average, and we have Stab$(A) \subset \Sigma(A)$ as a subgroup.

The subdivision of the phase space by flow-invariant regions restricts the possible symmetries of chaotic attractors. The possible symmetries on average of any chaotic attractor $A \subset \mathcal{C}$ with trivial instantaneous symmetries Stab$(A) = \{1\}$ are limited to subgroups of $\mathbb{Z}_N$ since they are contained in one of the invariant simplices ($\mathcal{C}$ or one of its images under the group action) with that symmetry. For $N = 4$, any chaotic attractor of this type must have trivial instantaneous symmetry. Thus, the possible symmetries on average are limited to subgroups of $\mathbb{Z}_4$, i.e., $\Sigma(A) \subset \mathbb{Z}_4$. In fact, we have found examples of chaotic attractors for each possible symmetry in systems of $N = 4$ and coupling functions given by (7) (Fig. 3). Note that this definition of attractor is somewhat restrictive—Milnor attractors may display a wider range of symmetries including different instantaneous symmetries at the same time.

Chaos for $N > 4$.—Analyzing the same region of parameter space for $N > 4$ yields attracting chaos in systems of $N = 5$ and $N = 7$ oscillators in large regions. Figure 4 shows an overlay of regions for three different N; regions are shaded where the Lyapunov exponent exceeds 0.01 and darker areas indicate that several N satisfy this condition. Clearly, there is a single coupling function for which attracting chaos is present for all $N = 4$, $N = 5$, and $N = 7$. Intriguingly, we did not find chaotic attractors for any $N \in \{6, 8, 9, \ldots, 13\}$ in the entire region of parameter space considered in Fig. 4.

The parametrization of the coupling function by a truncated Fourier series raises the question of how many Fourier components the coupling function needs to contain for chaos to occur. For $N = 5$ we also measured positive Lyapunov exponents when the coupling was chosen to be through the simpler coupling function $g(\varphi) = -0.2\cos(\varphi + \eta_1) - 0.04\cos(2\varphi - \eta_2)$ as in [9]. Hence, in dimension five, coupling functions with only two Fourier components suffice to generate chaotic dynamics whereas for $N = 4$, we did not find an example with less than four components.

From the above, it is clear that for systems of size $N = KM$ with $K \in \{4, 5, 7\}$ there are chaotic invariant sets lying in flow-invariant subspaces for coupling functions yielding positive $\lambda_{\max}$, cf. Fig. 4. For instance, for $K = 4$, these spaces are given by partitions $P = \{P_1, \ldots, P_4\}$ with $|P_j| = M$ for $j \in \{1, \ldots, 4\}$. For N large we similarly calculated positive maximal Lyapunov exponents for the system reduced to asymmetric 4-cluster states given by partitions $P = \{P_1, \ldots, P_4\}$ with $|P_1|/N = 1/4 + q$ and
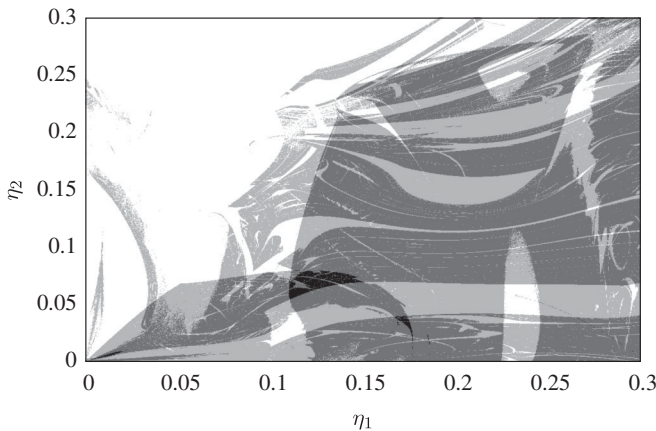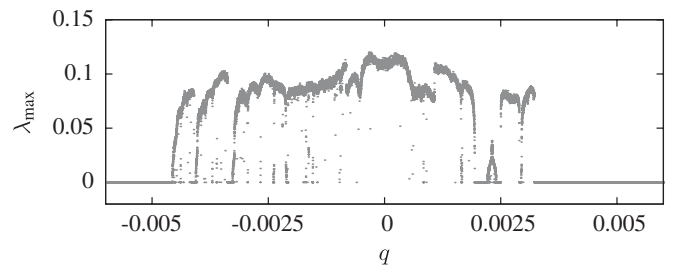

FIG. 4. Overlapping regions where the maximal Lyapunov exponent is greater than 0.01 for $N \in \{4, 5, 7\}$. The darker the color, the more N for which the condition holds. For parameter values around $(0.115, 0.06)$ there is a region where there is chaos for all these three N.

FIG. 5. Positive maximal Lyapunov exponents for asymmetric 4-cluster states for large systems, $N = KM + L(q, M) \gg 4$. Here q parametrizes the deviation from the symmetric cluster state and L is the corresponding integer dimension (coupling function (7) with $\eta_1 = 0.1104$ and $\eta_2 = 0.5586$).

$|P_j|/N = 1/4 - q/3$ for $j \in \{2, 3, 4\}$ as depicted in Fig. 5. However, these chaotic invariant sets in invariant subspaces close to the symmetric cluster state may be transversely repelling, possibly yielding nonchaotic long-term dynamics.

Discussion.—Inhomogeneities or asymmetries are thus not necessary for collective chaotic dynamics to appear in a system (1), and even chaotic order parameter fluctuations emerge in the presence of full $S_N \times T$ symmetry. We highlighted that for certain coupling functions chaotic attractors exist for several N. However, the regions in parameter space for which chaotic attractors exist vary drastically, cf. Fig. 4. For certain coupling functions there are chaotic invariant sets lying in flow-invariant subspaces that correspond to the symmetric and near-symmetric cluster states but these may not be transversely attracting. The question remains whether there are coupling functions giving rise to chaotic sets that are actually attracting for $N = 6$ and $N \geq 8$. Moreover, is there a "universal chaos function" in the sense that there is a coupling function for which there is some $N_0 \in \mathbb{N}$ such that there exists a chaotic attractors for all (or at least an infinite number of) $N > N_0$?

For coupling functions with only one Fourier component, finite-dimensional systems and the continuum limit are related; the dynamics for both finite N and in the continuum limit reduces to effectively two-dimensional dynamics [7,8] preventing the occurrence of chaotic trajectories. Is chaos possible in the continuum limit for more complicated coupling functions? If so, how would such a result relate to chaos in the finite-dimensional systems we have studied here?

For finite systems, attracting chaos in the system does not necessarily imply chaotic dynamics of the order parameter since there could be chaotic fluctuations, for example, in an "antiphase state." The converse, however, holds. Additionally, observed chaotic fluctuations of the order parameter cannot necessarily be traced back to inhomogeneity in the system (possibly through an experimental setup, cf. [19]) because, as shown above, even fully symmetric systems can support such dynamics. When considering the continuum limit, the problem of these implications becomes more subtle and will require further investigation.

The coupling function we considered above is written in terms of a truncated Fourier series. As discussed above, the number of Fourier components is relevant for the dynamics. An alternative approach would be to consider suitable piecewise linear functions. For $N = 4$, we find that systems with piecewise linear g also exhibit positive maximal Lyapunov exponents (not shown). Finding a suitable basis for the space of coupling functions might be a way to explain some of the dynamical features that were observed.

Coupled phase oscillators are a limit of weakly coupled limit cycle oscillators [2]. In globally coupled identical Ginzburg-Landau oscillator ensembles, chaotic dynamics can be observed [20,21]. However, it was thought that the amplitude degree of freedom is crucial for the emergence of such dynamics. Our results show that this is not the case and, moreover, suggest that chaotic mean field oscillations are also present in a large class of higher-dimensional symmetrically coupled limit cycle oscillators with a rich possible range of chaotic dynamics.

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